$\underbrace{\underbrace{9+10}_{19}+\underbrace{4+15}_{19}}_{38}+\underbrace{8+4}_{12}+\cdots$ 54.

50. $\frac{54}{100}$ → 13.5%

Math 201–Exam 1 (Fall 06)

B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1 and 2 of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-3)^{n+1}}{5^n}+\frac{2^{n-1}}{3^{n+2}}\right)$ (ii) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,(\ln n)^{100}}$ (iii) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2^n+n^2)}{e^n}$ (iv) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}-1}{n^{0.2}}$ div.

$\frac{1}{\sqrt{m}(\ln m)^{100}} > \frac{1}{(\ln m)^{100}} < \frac{1}{m}$

Problem 2 (answer on pages 3 and 4 of the booklet.)
(20 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{4^n\,n!\,n!}{(2n)!}(x-5)^n.$$

For what values of x does the series converge absolutely? Conditionally?

10

Problem 3 (answer on pages 5 and 6 of the booklet.)
(i) (8 pts) Prove that

$$\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\qquad |x|<1.$$

8

(ii) (4 pts) Let $f(x)=\arctan x$. Find $f^{(2007)}(0)$.
$\frac{f^{(2007)}(0)}{2007!}$ = coeff of x ... $x-\frac{x^3}{3}+\frac{x^5}{5}$
(iii) (7 pts) Find the Taylor polynomial $p_1(x)$ generated by $f(x)=\arctan x$ at the point $x=0$. Then use the alternating series estimation theorem to estimate the error resulting form the approximation

4

$$\arctan(-0.1)\approx p_1(-0.1).$$

Does $p_1(-0.1)$ tend to be too large or too small?
(iv) (6 pts) Use Taylor's theorem to estimate the error resulting from the approximation

$$\arctan(1/\sqrt{3})\approx p_1(1/\sqrt{3}).$$

Does $p_1(1/\sqrt{3})$ tend to be too large or too small? (Notice that $p_1(x)=p_2(x)$. You may need the fact that the third derivative of the function $f(x)=\arctan x$ is $f'''(x)=(6x^4+4x^2-2)/(1+x^2)^4$.)
(v) (4 pts) Find a power series expansion for the function

$$g(x)=\frac{\arctan x}{1-x}$$

4

about the point $x=0$. (It is enough to find the first four terms of the power series expansion.)

Problem 4 (answer on page 7 of the booklet.)
Consider the sequence

$$a_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

(i) (6 pts) Prove that $\ln(n+1)\le a_n\le 1+\ln n$ for all n.
(ii) (3 pts) Does $\lim_{n\to\infty}a_n$ exist? Why or why not?
(iii) (6 pts) What about $\lim_{n\to\infty}a_n/\ln n$?

$20+15+20+9=$
$29+35$
64

Wed, March 8 - 2006

6:00 - 7:00 p.m

Math 201 – Exam 1 (Spring 06)

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)
(8 pts each) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(i) $a_n = \dfrac{5^n + n^9 - 8n^2 - 3}{7n^9 + 2n + 5^{n+1}}$ (ii) $b_n = \dfrac{\sin(n! + 2n^3 - 7)}{\sqrt{n+3} - 1}$ (iii) $c_n = (3n+2)\arctan\left(\dfrac{1}{n}\right)$

Problem 2 (answer on pages 2 and 3 of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{2^n} + \frac{3^{n-1}}{5^{n+2}}\right)$ (ii) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(n+1)}$ (iii) $\displaystyle\sum_{n=2}^{\infty}\frac{\cos n}{n^n}$ (iv) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^{1.3}}$

Problem 3 (answer on pages 4 and 5 of the booklet.)
(20 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(e^{1/n} - 1)(x-4)^n.$$

For what values of x does the series converge absolutely? Conditionally?

Problem 4 (answer on page 6 of the booklet.)
(i) (5 pts) Find a power series expansion for $f(x) = \ln(1+x)$ about the point $x = 0$. Also find the Taylor polynomials $p_1(x)$ and $p_2(x)$ generated by f at the point $x = 0$.

(ii) (5 pts) Use the alternating series estimation theoren to estimate $\ln(1.1)$ with an error of magnitude less than 10^{-4}. Does your estimate tend to be too large or too small?

(iii) (5 pts) Use Taylor's theorem to show that

$$|f(x) - p_2(x)| \le \frac{8}{3}|x|^3, \quad \text{for} \quad |x| \le \frac{1}{2}.$$

$x - \frac{x^2}{2}$

(iv) (5 pts) Find

$$\lim_{n\to\infty} e^{4n}\left(1 - \frac{4}{n}\right)^{n^2}$$

Math 201 – Exam 1 (Spring 07) B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\sum_{n=1}^{\infty}\left(\frac{(-2)^{n-1}}{3^n}+\frac{1}{5^{n+2}}\right)$ (ii) $\sum_{n=1}^{\infty}\left(\frac{n-3}{n+1}\right)^n$ (iii) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ (iv) $\sum_{n=1}^{\infty}\ln(1+\frac{1}{n^3})$

Problem 2 (answer on page **3** of the booklet.)
(15 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(x-3)^n.$$

For what values of x does the series converge absolutely? Conditionally?

Problem 3 (answer on pages 4 and 5 of the booklet.)
(i) (10 pts) Find a power series expansion for $f(x)=\sqrt{1+x}$ about the point $x=0$. Also find the Taylor polynomials $p_1(x)$ and $p_2(x)$ generated by f at the point $x=0$.

(ii) (5 pts) Express $\int\sqrt{1+x^4}\,dx$ as a power series.

(iii) (5 pts) Approximate $\sqrt{1.01}$ by $p_2(?)$ and use the alternating series estimation theorem to estimate the resulting error.

(iv) (9 pts) Approximate $\sqrt{0.85}$ by $p_1(?)$ and use Taylor's theorem to estimate the resulting error.

Problem 4 (answer on page **6** of the booklet.)
(a) (10 pts) Find the Fourier series of the function $f(x)=x$, $0\le x\le 2\pi$.

(b) (5 pts) Find the sum of the series in part (a) for $0\le x\le 2\pi$.

(c) (5 pts) Use the result of part (b) to find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}.$$

$\frac{2}{3^2}+\frac{4}{3^3}$ $\frac{2}{3}$ $\frac{1}{5^3}+\frac{1}{5^4}+$

$0, \frac{\pi}{2}$

Math 201–Exam 2 (Fall 04) Prof. B. Shayya and Dr. H. Yamani

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)

(15 pts) Find the domain and range of the function $f(x, y, z) = 3/(x^2 + y^2 + z^2 - 9)$ and identify its level surfaces. Determine if the domain of f is an open region, a closed region, or neither. Also, decide if the domain is bounded or unbounded.

Problem 2 (answer on page 2 of the booklet.)

(a) (7 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x}\,y}{\sqrt{y^2 + x^2}}$$

exist? Why or why not?

(b) (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x}\,y}{\sqrt{y^2 - x^2}}?$$

Problem 3 (answer on pages 3, 4, and 5 of the booklet.)

Suppose that the derivative of the function $f(x, y, z)$ at the point $(1, 1, 1)$ is greatest in the direction of $A = 6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, and that in this direction the value of the derivative is $\sqrt{6}$. Also suppose that

$$f(3, 0, -1) = 1, \quad \nabla f(3, 0, -1) = 3\mathbf{i}-\mathbf{j}+5\mathbf{k}, \quad \nabla f(3, 2, 1) = 6\mathbf{i}-2\mathbf{j}+\mathbf{k} \quad \text{and} \quad \nabla f(0, -1, 1) = \mathbf{i}+\mathbf{j}+\mathbf{k}.$$

(a) (5 pts) Find the derivative of f at the point $(3, 2, 1)$ in the direction of $\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$.

(b) (10 pts) Find $\nabla f(1, 1, 1)$.

(c) (4 pts) Is there a unit vector u such that $D_u f(3, 0, -1) = 6$? Give reasons for your answer.

(d) (6 pts) Find the line normal to the surface $f(x, y, z) = 1$ at the point $(3, 0, -1)$.

(e) (10 pts) Let

$$x = \alpha, \qquad y = \beta - 2, \qquad z = \beta - \alpha, \quad \text{and} \quad \omega = f(x, y, z).$$

Find $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$ at the point $(\alpha, \beta) = (3, 2)$.

(f) (10 pts) Let $\omega = \omega(\alpha, \beta)$ be as in part (e). Find a plane tangent to the surface

$$\omega(\alpha, \beta) = 2\gamma^2 - 1$$

in the $\alpha\beta\gamma$-space. (Hint. Start by finding a point $(\alpha, \beta, \gamma) = (?, ?, ?)$ on the surface.)

Problem 4 (answer on page 6 of the booklet.)

(a) (9 pts) Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < -1, \\ 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{if } 1 < x < \pi. \end{cases}$$

(Notice that $L = \pi$, not 1.)

(b) (10 pts) Find the sum of the series in part (a) for $-\pi \le x \le \pi$.

(c) (6 pts) Use the result of part (b) to find

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\sin(2n)}{n}.$$

Math 201 – Exam 2 (Spring 04) B. Shayya

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)

(a) (4 pts) Write the point $(x, y) = (\sqrt{2}, \sqrt{2})$ in polar coordinates.

(b) (8 pts) Find all polar coordinate representations of the point $(r, \theta) = (5, \pi/3)$.

(c) (6 pts) Graph the polar curve $r = 1 + 2\cos\theta$ for $0 \le \theta \le \pi$.

Problem 2 (answer on page 2 of the booklet.)

(16 pts) Find the tangent plane and normal line of the surface $z = x^2 + y^2$ at the point $(1, 2, 3)$.

Problem 3 (answer on page 3 of the booklet.)

(16 pts) Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 2x^3 + 6x^2 - 4y^3 + 3y^2.$$

Problem 4 (answer on page 4 of the booklet.)

(a) (6 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^5}{x^6 + x^2y^4}$$

exist? Why or why not?

(b) (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^5}{3y - 2x}?$$

Problem 5 (answer on pages 5 and 6 of the booklet.)

Suppose that the derivative of the function $f(x, y, z)$ at the point $(1, 2, 1)$ is greatest in the direction of $A = \mathbf{i} - \mathbf{j} + \sqrt{2}\mathbf{k}$, and that in this direction the value of the derivative is 12. Also suppose that

$f(1, 1, 1) = 22$, $\nabla f(1, 1, 1) = \mathbf{i}+2\mathbf{j}-2\mathbf{k}$, $\nabla f(3, 5, -3) = 2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $\nabla f(0, -1, 1) = \mathbf{i}+\mathbf{j}+\mathbf{k}$.

(a) (9 pts) Find $\nabla f(1, 2, 1)$.

(b) (3 pts) Is there a unit vector u such that $D_u f(3, 5, -3) = -4$? Give reasons for your answer.

(c) (12 pts) Approximate $f(1.1, 1.05, 0.95)$

(d) (6 pts) Let

$$x = r + s, \quad y = r + 2s, \quad z = r - s^2, \quad \text{and} \quad w = f(x, y, z).$$

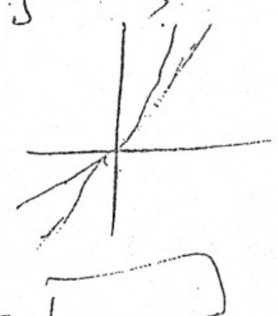
Find $\partial w/\partial s$ at the point $(r, s) = (1, 2)$.

(e) (6 pts) Let $w = w(r, s)$ be as in part (d). Find a line tangent to the curve

$$w(r, s) = 22(r + s)$$

in the rs-plane. (Hint. Start by finding a point $(r, s) = (?, ?)$ on the curve.)

$y = \frac{2x}{3}$



$x = 1$

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Problem 1 (answer on page 1 of the booklet.)
(16 pts) Find the tangent plane and normal line of the surface $z = x + x^2 + y^2$ at the point $(-1, 2, 4)$.

Problem 2 (answer on page 2 of the booklet.)
(a) (11 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^5 y + x^2 y^2}{x^4 + y^2}$$

exist? Why or why not?
(b) (5 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^4 + y^4}?$$

(c) (6 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^4 + y}?$$

Problem 3 (answer on pages 3, 4 and 5 of the booklet.)
(a) (14 pts) Find the domain and range of the function

$$f(x,y) = \begin{cases} \sqrt{1 - x^2 - y^2} & \text{if } y \geq 0, \\ \frac{2}{\sqrt{1-4x^2-4y^2}} & \text{if } y < 0. \end{cases}$$

Determine if the domain of f is an open region, a closed region, or neither. Also, decide if the domain is bounded or unbounded.
(b) (6 pts) Find the level curves $f(x,y) = 3/4$ and $f(x,y) = 3$.
(c) (15 pts) Let

$$x = \frac{r + 3s}{6 + r + s}, \qquad y = \frac{r + \ln s}{1 + r}, \quad \text{and} \quad w = f(x,y).$$

Find $\partial w/\partial r$ and $\partial w/\partial s$ at the point $(r,s) = (1,1)$. Also, find the derivative of w at the point $(r,s) = (1,1)$ in the direction of $\mathbf{i} + \mathbf{j}$.
(d) (7 pts) Let $w = w(r,s)$ be as in part (c). Find a line tangent to the curve

$$2\sqrt{2}\, w(r,s) = r^5 + s^3$$

in the rs-plane.

Problem 4 (answer on page 6 of the booklet.)
(a) (10 pts) Find the Fourier series of the function

$$f(x,y) = \begin{cases} x^2 & \text{if } 0 \leq x \leq \pi, \\ 0 & \text{if } \pi < x \leq 2\pi. \end{cases}$$

(See back page for relevant integrals.)
(b) (5 pts) Find the sum of the series in part (a) for $0 \leq x \leq 2\pi$.
(c) (5 pts) Use the result of part (b) to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Needed

$$\int x^2 \cos(nx)\,dx = \frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3}$$

$$\int x^2 \sin(nx)\,dx = -\frac{x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2\cos nx}{n^3}$$

Math 201 – Exam 2 (Spring 07) B. Shayya

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- Unjustified answers will receive little or no credit.

Problem 1 (answer on page **1** of the booklet.)
(13 pts) Find the domain and range of the function

$$f(x,y,z) = \frac{2}{1-x^2-y^2-z^2}.$$

Determine if the domain of f is an open region, a closed region, or neither. Also, decide if the domain is bounded or unbounded.

Problem 2 (answer on page **2** of the booklet.)
(a) (11 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3 \sin e^x}{x^8+y^2}$$

exist? Why or why not?
(b) (9 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3 \sin e^x}{x^8-y^2}?$$

Problem 3 (answer on pages **3** and **4** of the booklet.)
Consider the function $f(x,y) = y + x^2 + y^2$.
(a) (12 pts) Find the tangent plane and normal line of the surface $z = f(x,y)$ at the point $(2,-1,4)$.
(b) (5 pts) Suppose the equation $f(x,y) - z^3 = \sin z$ defines z implicitly as a function of (x,y). Find $\partial z/\partial y$.
(c) (9 pts) Prove that $f(x,y)$ is differentiable everywhere in the plane.

line $3\sqrt{6}$

9

Problem 4 (answer on pages **5** and **6** of the booklet.)
Suppose that the derivative of the function $f(x,y,z)$ at the point $(1,1,1)$ is greatest in the direction of $A = 6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, and that in this direction the value of the derivative is $\sqrt{6}$. Also suppose that

$$f(3,0,0) = 1, \quad \nabla f(3,0,0) = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}, \quad f(3,2,1) = 3 \quad \text{and} \quad \nabla f(3,2,1) = 6\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

(a) (6 pts) Find the derivative of f at the point $(3,2,1)$ in the direction of $\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$.
(b) (10 pts) Find $\nabla f(1,1,1)$.
(c) (5 pts) Is there a unit vector u such that $D_u f(3,2,1) = -7$? Give reasons for your answer.
(d) (10 pts) Let

$$x = r^2 + s, \quad y = rs, \quad z = \frac{2r}{s}, \quad \text{and} \quad w = f(x,y,z).$$

Find $\partial w/\partial s$ at the point $(r,s) = (1,2)$.

$-\frac{2r}{s^2}$

(e) (10 pts) Let $w = w(r,s)$ be as in part (c). Find a plane tangent to the surface

$$\frac{2}{1-t} + 3 = w(r,s)$$

in the rst-space. (Hint. Start by finding a point (r_0, s_0, t_0) on the surface.)

line $\frac{2r}{s}$ $-\frac{2r}{s^2}$

$6(1) - 2(r) + 1\left(-\frac{2r}{s^2}\right)$

$1 - 2 - \frac{2}{4}$

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Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(a) (8 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(0,0,0)}^{(1,0,2)} (e^x \cos y + yz)\, dx + (xz - e^x \sin y)\, dy + (xy + z)\, dz$$

(b) (10 pts) Find the counterclockwise circulation and outward flux of the field $F(x, y) = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$ around and across the cardioid $r = 2(1 + \cos\theta)$.

Problem 2 (answer on pages 4 and 5 of the booklet.)

(a) (8 pts) Find the volume of the region in the first octant that is bounded by the coordinate planes, the plane $2x + 3z - 12 = 0$, and the surface $y = \frac{1}{2}z^2$.

(b) (10 pts) Let D be the smaller region cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/6$. Evaluate $\iiint_D z\, dV$.

Problem 3 (answer on pages 6 and 7 of the booklet.)

(a) (8 pts) Find the points on the sphere $x^2 + y^2 + z^2 = 30$ where $f(x, y, z) = x - 2y + 5z$ has its maximum and minimum values.

(b) (10 pts) Suppose $f(x, y, z)$ is a differentiable function with $\nabla f(0, 0, 0) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\nabla f(2, 0, 0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Let

$$x = 2\sqrt{1 - r^2 - s^2}, \qquad y = 3r, \qquad z = 5s, \qquad \text{and} \qquad w = f(x, y, z).$$

Find $\partial w/\partial r$ and $\partial w/\partial s$ at the point $(r, s) = (0, 0)$.

Problem 4 (answer on page 8 of the booklet.)

(12 pts) Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the <u>surface</u> cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Problem 5 (answer on pages 9, 10, and 11 of the booklet.)

(a) (6 pts) Find the Fourier series of the function $f(x) = x$ on the interval $-\pi < x < \pi$.

(b) (6 pts) Use the series in part (a) to find $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$.

(c) (6 pts) Determine whether the series $\sum_{n=1}^{\infty} (\ln n)\left(e^{1/n} - 1\right)^{13}$ converges or diverges.

Problem 6 (answer on pages 12 and 13 of the booklet.)

(16 pts) Let D be the disk $x^2 + y^2 \leq 3/4$. Use the transformation

$$x = u\sqrt{1 - \frac{u^2}{4} - \frac{v^2}{2}}, \qquad y = v\sqrt{1 - \frac{v^2}{4}}$$

to rewrite

$$\iint_D 8xy\sqrt{1 - x^2 - y^2}\, dx\, dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G. (Hint: $\sqrt{1 - x^2 - y^2} = 1 - \frac{1}{2}(u^2 + v^2)$.)

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Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(a) (8 pts) Find the line integral of $f(x,y,z) = x+y+z$ over the straight-line segment from (1, to $(0,-1,1)$.

(b) (8 pts) Find the work done by $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ over the curve $C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} +$ $0 \le t \le 1$.

(c) (9 pts) Show that the differential form in the following integral is exact, then evaluate integral.

$$\int_{(1,1,0)}^{(5,0,8)} (e^y + yz)\,dx + (xe^y + xz)\,dy + (xy)\,dz$$

Problem 2 (answer on page 4 of the booklet.)

(10 pts) Suppose that the derivative of the function $f(x,y,z)$ at the point $P(1,5,2)$ is greates the direction of $\mathbf{A} = 3\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$. Also suppose that the value of the derivative of f at P in direction of $\mathbf{B} = \mathbf{i} + \mathbf{j} - \sqrt{2}\mathbf{k}$ is 10. Find $\nabla f(P)$.

Problem 3 (answer on pages 5 and 6 of the booklet.)

(15 pts) Find the maximum value of $f(x,y,z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 4 (answer on pages 7, 8, and 9 of the booklet.)

Let D be the smaller region cut from the sphere $x^2 + y^2 + z^2 = 4$ by the plane $z = \sqrt{3}$.

(a) (6 pts) Set up the triple integral in spherical coordinates that give the volume of D using order of integration $d\rho\, d\phi\, d\theta$.

(b) (8 pts) Set up the triple integrals in spherical coordinates that give the volume of D using order of integration $d\phi\, d\rho\, d\theta$.

(c) (4 pts) Use the integral of part (a) to find the volume of D.

Problem 5 (answer on pages 10 and 11 of the booklet.)

(10 pts) Use the transformation

$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2}, \qquad w = \frac{z}{3}$$

to rewrite

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz$$

as an integral over an appropriate region G in the uvw-space. Then evaluate the uvw-integral over

Problem 6 (answer on pages 12 and 13 of the booklet.)

(4 pts each) Which of the following series converge, and which diverge? Find the sum of e convergent series.

$$(i)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}+5} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{\ln(1+e^n)}{n^2} \qquad (iii)\ \sum_{n=2}^{\infty} \frac{3^n}{n!}$$

Problem 7 (answer on pages 14 and 15 of the booklet.)

(a) (5 pts) Use that $(\arctan x)' = 1/(1+x^2)$ to find a power series expansion for $\arctan x$ about point $x = 0$.

(b) (5 pts) For what values of x can we replace $\arctan x$ by $x - x^3/3$ with an error of magnitude greater than 2×10^{-16}?

(A1)

(1)

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Problem 1 (answer on pages **1** and **2** of the booklet.)
(24 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(5,0,9)}^{(1,\pi,0)} (2x\cos y + yz)\,dx + (xz - x^2\sin y)\,dy + (xy)\,dz$$

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Problem 2 (answer on pages **3** and **4** of the booklet.)
(24 pts) Find the maximum and minimum values of $f(x,y,z) = xyz$ on the sphere $x^2+y^2+z^2=1$.

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Problem 3 (answer on pages **5** and **6** of the booklet.)
Let D be the region bounded below by the plane $z=0$, above by the sphere $x^2+y^2+z^2=4$, and on the sides by the cylinder $x^2+y^2=1$.
(i) **(8 pts)** Set up the triple integrals in cylindrical coordinates that give the volume of D using the order of integration $dz\,r\,dr\,d\theta$. Then find the volume of D.
(ii) **(6 pts)** Set up the limits of integration for evaluating the integral of a function $f(x,y,z)$ over D as an iterated triple integral in the order $dy\,dz\,dx$.
(iii) **(12 pts)** Set up the triple integrals in spherical coordinates that give the volume of D using the order of integration $d\phi\,d\rho\,d\theta$.

Problem 4 (answer on pages **7** and **8** of the booklet.)
(25 pts) Integrate $g(x,y,z) = z$ over the surface of the prism cut from the first octant by the planes $z=x$, $z=2-x$, and $y=2$.

Problem 5 (answer on pages **9**, **10**, and **11** of the booklet.)
Let S be the cone $z = 1-\sqrt{x^2+y^2}$, $0 \le z \le 1$, and let C be its base (i.e. C is the unit circle in the xy-plane). Find the counterclockwise circulation of the field

$$F(x,y,z) = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

around C
(a) **(12 pts)** directly,
(b) **(8 pts)** using Green's theorem, and
(c) **(14 pts)** using Stokes' theorem (i.e. by evaluating the flux of curl F outward through S).

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Problem 6 (answer on pages **12** and **13** of the booklet.)
(25 pts) Let R be the region in the xy-plane bounded by the lines $y=0$, $y=x$, $x+y=4$, and $x+y=9$. Use the transformation

$$x = uv, \qquad y = (1-u)v$$

to rewrite

$$\int\!\!\int_R \frac{1}{\sqrt{x+y}}\,dx\,dy$$

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Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(i) (8 pts) Find the line integral of $f(x, y, z) = xy+y+z$ over the straight-line segment from $(0, 0, 2)$ to $(2, 1, 0)$.

(ii) (17 pts) Let R be the region in the first quadrant bounded by the x-axis, the y-axis, and the line $x+y=1$. Let C be the boundary of R traced counterclockwise. Find

$$\int_C y^2 dx + 3x^2 dy$$

(a) directly, and (b) using Green's theorem.

Problem 2 (answer on page 4 of the booklet.)

(8 pts) Find the surface area of the upper portion of the cylinder $x^2+z^2=1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$. (Hint. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$.)

Problem 3 (answer on pages 5 and 6 of the booklet.)

(15 pts) Find the maximum and minimum values of $f(x, y, z) = x + 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 25$.

Problem 4 (answer on pages 7 and 8 of the booklet.)

Let D be the region that lies inside the sphere $\rho = 2$ and between the cones $\phi = \pi/6$ and $\phi = \pi/4$.

(i) (10 pts) Set up the triple integral in spherical coordinates that gives the volume of D. Then evaluate the integral.

(ii) (10 pts) Set up, but do not evaluate, the triple integrals in cylindrical coordinates that give the volume of D.

Problem 5 (answer on pages 9 and 10 of the booklet.)

(15 pts) Use the transformation

$$u = x, \qquad v = xy, \qquad w = 3z$$

to rewrite

$$\int_0^1 \int_1^2 \int_0^{2/x} (x^2y + 3xyz)\, dy\, dx\, dz$$

as an integral over an appropriate region G in the uvw-space. Then evaluate the uvw-integral over G.

Problem 6 (answer on page 11 of the booklet.)

(8 pts) Use the fact that $(\arctan x)' = 1/(1+x^2)$ to find a power series expansion for $\arctan x$ about the point $x = 0$. Then decide whether $\sum_{n=1}^{\infty} n^{-0.1} \arctan(1/n)$ converges or diverges.

Problem 7 (answer on pages 12 and 13 of the booklet.)

(i) (5 pts) Find the Fourier series expansion of the function $f(x) = x$ over the interval $-\pi < x < \pi$. (Hint. Since f is an odd function, its Fourier series is of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$.)

(ii) (2 pts) Find the sum of the series in part (i) for $-\pi \leq x \leq \pi$.

(iii) (2 pts) Use the result of part (ii) to show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

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Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(i) (8 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(2,0,1)}^{(6,7,0)} \left(\sin(yz) - \frac{2xz}{1+x^2}\right) dx + xz\cos(yz)\,dy + \left(xy\cos(yz) - \ln(1+x^2)\right) dz$$

(ii) Find the flux of the field $F(x,y) = 2x\mathbf{i} - y\mathbf{j}$ across the region in the first quadrant bounded by the coordinate axes and the curve $y = x - x^2$, $0 \le x \le 1$,
(a) (8 pts) directly
(b) (6 pts) using Green's theorem.

(iii) (3 pts) Suppose $F(x,y)$ is a vector field on an open, connected, and simply connected region R in the xy-plane. Recall that the (two-dimensional) *curl test* says that

$$F \text{ is conservative in } R \Leftrightarrow \text{curl } F = 0 \text{ everywhere in } R.$$

Use Green's theorem to prove the $\Leftarrow$ implication.

Problem 2 (answer on pages 4, 5, and 6 of the booklet.)

(i) Let D be the region in space bounded below by the plane $z = 0$, above by the sphere $x^2+y^2+z^2 = 4$, and on the sides by the cylinder $x^2+y^2 = 1$. Set up (but do not evaluate) the iterated integrals in spherical coordinates that give the volume of D in the order
(a) (7 pts) $d\rho\, d\phi\, d\theta$
(b) (7 pts) $d\phi\, d\rho\, d\theta$.

(ii) (4 pts) Let D as in part (i). Set up (but do not evaluate) the iterated integral in cylindrical coordinates that give the volume of D in the order $dz\, dr\, d\theta$.

(iii) (8 pts) Let R be the region in space bounded by the planes $z = y$ and $y = 1$ and the surface $z = x^2$. Evaluate

$$\iiint_R \frac{2}{(x^2-1)^2}\, dV.$$

Problem 3 (answer on pages 7 and 8 of the booklet.)

(i) (8 pts) Find all local maxima, local minima, and saddle points of the function $f(x,y) = x^3 - y^3 - 2xy + 6$.

(ii) (7 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = 2x + 2y + 2z$ on the sphere $x^2+y^2+z^2 = 3$.

Problem 4 (answer on pages 9 and 10 of the booklet.)

(14 pts) Let R be the region in the xy-plane bounded by the lines $y = 0$, $y = 2x - 2$, $y = 4$, and $y = 2x$. Use the transformation

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2}$$

to rewrite

$$\iint_R \frac{2x - y}{2}\, dA$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 5 (answer on page 11 of the booklet.)

(10 pts) Find the linearization $L(x, y)$ of the function $f(x, y) = x^2 - 3xy + 5$ at the point $(2, 1)$. Then estimate the error that results when one approximates $f(x, y)$ by $L(x, y)$ over the rectangle

$$R: \quad |x - 2| \leq 0.1, \quad |y - 1| \leq 0.1.$$

Problem 6 (answer on pages 12 and 13 of the booklet.)

(i) (3 pts) Does $\sum_{n=1}^{\infty} e^{1-n}$ converge? Why or why not?

(ii) (3 pts) What about $\sum_{n=1}^{\infty} e^{-1/n^2}$?

(iii) (4 pts) What about $\sum_{n=1}^{\infty} \left(1 - e^{-1/n^2}\right)$?

$$\frac{u_{n+1}}{u_n}$$

$$< 0$$

$$\frac{e^{1-n+1}}{e^{1-n}} = \frac{e^{-n}}{e^{1-n}} = \frac{e^{-n}}{e \cdot e^{-n}} = \frac{1}{e}$$

$$e^{-\frac{1}{n^2}} \qquad e^{\frac{1}{e^{-n}}}$$

$$\frac{e}{e^{n}}$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 7 (answer on page **14** of the booklet.)
(6 pts each) Determine which of the following series converge, and which diverge.

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(a) $\sum_{n=1}^{\infty} \sqrt{n} \ln\left(1 + \frac{1}{n^{2.1}}\right)$ (b) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{1.2}}$ (c) $\sum_{n=1}^{\infty} n(\sqrt[n]{n} - 1)$

Problem 8 (answer on pages 15 and 16 of the booklet.)
(i) (6 pts) Use Taylor's theorem to prove that

6

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

(ii) (6 pts) Approximate

6

$$\int_0^{0.1} e^{-x^2}\,dx$$

with an error of magnitude less than 10^{-5}.

(iii) (6 pts) Show that

6

$$\int_0^{\infty} e^{-\pi x^2}\,dx = \frac{1}{2}.$$

(Hint. If $I = \int_0^\infty e^{-\pi x^2}dx$, then $I^2 = \int_0^\infty \int_0^\infty e^{-\pi(x^2+y^2)}dxdy$.)

(iv) (6 pts) Let E be the error resulting from the approximation

$$\int_0^{100} e^{-\pi x^2}\,dx \approx \frac{1}{2}.$$

Show that

$$|E| < \frac{e^{-5000\pi}}{2}.$$

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Problem 1 (answer on pages 1 and 2 of the booklet.)
(a) (10 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(1,-2,5)}^{(2,-2,3)} (2xy - 3z^2)\,dx + (x^2 - 4yz)\,dy - 2(y^2 + 3xz)\,dz$$

(b) (10 pts) Find the counterclockwise circulation and outward flux of the field

$$F(x, y) = (y^2 - x^2)\mathrm{i} + (x^2 + y^2)\mathrm{j}$$

Green

around and across the triangle bounded by the lines $y = 0$, $x = 3$, and $y = x$.

Problem 2 (answer on pages 3, 4, and 5 of the booklet.)
(a) (11 pts) Find the volume of the region in space that is bounded by the planes $x = 0$, $y = 0$, $z = x$, $x + z = 12$ and the paraboloid $y = x^2 + z^2$.

(b) (11 pts) Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

(c) (11 pts) Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Problem 3 (answer on pages 6 and 7 of the booklet.)
(12 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Problem 4 (answer on pages 8 and 9 of the booklet.)
(10 pts) Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 12x^2 + 12y^2 + (x + y)^3.$$

Problem 5 (answer on pages 10 and 11 of the booklet.)
(15 pts) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation

$$x = \frac{u}{v}, \qquad y = uv$$

to rewrite

$$\int\int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 6 (answer on pages 12 and 13 of the booklet.)
(a) (3 pts) Does $\sum_{n=1}^{\infty}(1 + \frac{1}{n^2})$ converge? Why or why not?

(b) (7 pts) What about $\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n^2})$?

Math 201–Final Exam (Spring 07) B. Shayya

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- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(20 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = e^{x+y+z}$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 2 (answer on pages **3**, **4**, and **5** of the booklet.)
(i) (13 pts) Let V be the volume of the region in space that is bounded below by the xy-plane, on the sides by the cylinder $y = x^2$, and above by the plane $y + z = 1$. Write V as an iterated triple integral in the order $dydxdz$. Then find V.

(ii) Let D be the region in space bounded below by the plane $z = 1$, on the sides by the cylinder $x^2 + y^2 = 1$, and above by the sphere $x^2 + y^2 + z^2 = 4$. Suppose $f(x, y, z)$ is a continuous function on D and let

$$I = \iiint_D f(x, y, z)\,dV.$$

(a) (10 pts) Write I as an iterated triple integral(s) in cylindrical coordinates in the order $dzdrd\theta$.
(b) (13 pts) Write I as an iterated triple integral(s) in spherical coordinates in the order $d\rho d\phi d\theta$.
(c) (13 pts) Write I as an iterated triple integral(s) in spherical coordinates in the order $d\phi d\rho d\theta$.

Problem 3 (answer on pages **6**, **7**, and **8** of the booklet.)
(i) (18 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(1,-1,1)}^{(3,3,-1)} (3z^2 - 2xy)\,dx + (4yz - x^2)\,dy + (2y^2 + 6xz)\,dz$$

(ii) Let R be the region in the first quadrant that is bounded by the x-axis, the line $x = 1$, and the curve $y = x^2$. Find the outward flux of the field $F(x, y) = (x/2)\mathbf{i} + (y/2)\mathbf{j}$ across the boundary of R
(a) (13 pts) directly
(b) (8 pts) using Green's theorem.